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In this note we show how Ward identities may be derived for a quantum field theory dual of a string theory using the AdS/CFT correspondence. In particular associated with any gauge symmetry of the bulk supergravity theory there is a corresponding constraint equation. Writing this constraint in Hamilton-Jacobi form gives a generating functional for Ward identities in the dual QFT. We illustrate the method by considering various examples.

I. INTRODUCTION

The development of the AdS/CFT correspondence [1–3], see [4] for a review, has shown that the “radial” coordinate of anti-de Sitter spacetime, by which we mean a coordinate transverse to the timelike boundary, corresponds in the dual CFT to an energy scale. In particular the UV/IR relations discussed in [5,6] illustrate how large energies in the CFT correspond to being near the boundary of AdS, and conversely how small energies in the CFT correspond to being far from the boundary of AdS. This has led to the realization that “radial” evolution in the bulk corresponds to renormalization group flow in the boundary, i.e., that the bulk equations of motion are the renormalization group equations of the boundary correlators. This idea has been elucidated in various cases involving the duality between type IIB string theory on $AdS_5 \times S^5$ and 4-dimensional $\mathcal{N} = 4$ SYM, [7–15]. There have further been more general analyses given in [16–18]. In particular de Boer, Verlinde, and Verlinde have given an explicit construction of the renormalization group equations from the equations of motion for a set of scalar fields coupled to gravity. Their method consists of first rewriting the equations of motion for the coupled system in Hamilton-Jacobi form using a radial coordinate as the evolution parameter. Diffeomorphism invariance gives rise to the Hamiltonian and vector constraints, the former of which [17] use prominently to construct the dual QFT renormalization group equations. Their analysis however makes no use of the vector constraints, so it is natural to wonder what role it plays in the grand scheme of things. We show below that it, and more generally any constraint arising from a gauge symmetry, gives rise to Ward identities in the dual QFT. Conversely this means that in trying to reconstruct the bulk spacetime from the boundary QFT (see, eg., [19–21]), Ward identities in the QFT will play an important role in constructing the bulk constraint equations.

The subject of Ward identities in AdS/CFT has arisen as a useful check on the correspondence, i.e., one may compute CFT correlators from supergravity on anti-de Sitter spacetime and check that they satisfy the required Ward identities. Such checks have been carried out in various cases. For the duality between 4d, $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and type IIB string theory on $AdS_5 \times S^5$ a check on the Ward identity of the R -current correlators $\langle J_\mu^a(x) J_\nu^b(y) J_\rho^c(z) \rangle$ of the SYM theory has been carried out in [22,23]. [22] also showed that the dual QFT correlator $\langle J_\mu^a(x) \mathcal{O}^I(y) \mathcal{O}^J(z) \rangle$ (where $\mathcal{O}^I(x)$ is a gauge invariant composite scalar) satisfies the appropriate Ward identity, with similar results for spinors replacing the above scalars reported in [24]. One also has Ward identities involving the stress-tensor of the dual CFT and indeed [25] showed that correlators of the form $\langle T_{\mu\nu}(x) \mathcal{O}(y) \mathcal{O}(z) \rangle$ computed in the supergravity approximation satisfy the appropriate identity. Our purpose in this note is to show that indeed these Ward identities, as well as a more general set of identities, follow quite simply from the supergravity constraint equations. Specifically, corresponding to any gauge symmetry on the supergravity side of the duality is a constraint equation. Rewriting this constraint equation in Hamilton-Jacobi form as a functional of the boundary values of the supergravity fields, taking the appropriate functional derivatives of the resulting equation with respect to the boundary values of the fields, and evaluating on the appropriate background yields precisely the dual QFT Ward identities.

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To begin, let's review the prescription [2,3] for generating correlators of operators in the dual quantum field theory from supergravity on an asymptotic AdS_{d+1} space. Following [17] we work with the Euclidean signature case and assume that the manifold is topologically an $d+1$ -ball as is Euclidean AdS_{d+1} , but that geometrically the space need only be asymptotic to AdS_{d+1} . We write the metric in ADM form, appropriate for radial evolution, as¹

$$ds^2 = N^2 dr^2 + h_{ij}(x, r)(dx^i + N^i dr)(dx^j + N^j dr). \quad (2.1)$$

In particular we will eventually choose the “radial” gauge conditions $N = 1$ and $N^i = 0$, and furthermore take $h_{ij}(r, x) = \delta_{ij} \exp(-2\lambda(r))$ as in the “kink” solutions² considered in [7–15], but for now leave the metric in the general form (2.1). The prescription for generating correlators associates to each field ϕ_i in the string theory spectrum a corresponding local operator \mathcal{O}^i in the QFT such that the relation (in Euclidean signature)

$$Z[\phi_{i,0}] = \left\langle e^{\int_{\partial} \phi_{i,0} \mathcal{O}^i} \right\rangle \quad (2.2)$$

holds, where the left-hand-side is the string theory partition function with the background fields ϕ_i turned on with boundary values $\phi_{i,0}$. In the large N , large 't Hooft coupling limit, the partition function can be approximated by $\exp(-S_{eff}[\phi_{i,0}])$ where S_{eff} is the low energy supergravity effective action evaluated on the solutions to it's equations of motion subject to the boundary conditions $\phi_i|_{\partial} = \phi_{i,0}$. To regulate the action $S_{eff}[\phi_{i,0}]$ we take the boundary at finite r . On the right-hand-side of (2.2) the expectation value of the given exponential is taken in the dual quantum field theory, with $\phi_{i,0}$ acting as a source for the QFT operator \mathcal{O}^i . In particular the boundary value of a bulk scalar, spinor, massless vector, massless gravitino, and massless graviton acts respectively as the source for a scalar, spinor, conserved current, conserved SUSY current, and stress tensor in the dual QFT. To construct correlators for the QFT fields $\mathcal{O}^i(x)$ now, one need just functionally differentiate the above relation (2.2) with respect to the source $\phi_{i,0}$ the appropriate number of times, eg.,

$$\langle \mathcal{O}^{i_1}(x_1) \cdots \mathcal{O}^{i_n}(x_n) \rangle = -\frac{\delta}{\delta \phi_{i_1,0}(x_1)} \cdots \frac{\delta}{\delta \phi_{i_n,0}(x_n)} S_{eff}[\phi_{i,0}]|_{\phi_{i,0}=0}. \quad (2.3)$$

A. Vector case:

According to the above dictionary between bulk and boundary fields, to derive QFT Ward identities the dual supergravity theory must contain some massless vectors, gravitinos, and a graviton. For simplicity we shall ignore the gravitino, commenting on it later, and concentrate on the other two fields. In particular let's first consider a real scalar field minimally coupled to a non-Abelian vector field with gauge group $SO(N)$, and take the scalars in the representation t_a^{IJ} where the matrices t_a^{IJ} are imaginary and antisymmetric. There are of course more relevant systems to consider as far as the AdS/CFT duality is concerned, eg., 5d gauged $\mathcal{N} = 8$ supergravity, but our purpose here is only to illustrate the method of generating Ward identities in the dual QFT from the supergravity description of the theory, and with this goal in mind the simple system we consider here, and in the next section, suffices. The action for this system in a curved spacetime is

$$S_{\phi} + S_A = \int d^{d+1}x \sqrt{g} (\tilde{\nabla}_{\mu} \phi^I \tilde{\nabla}^{\mu} \phi^I + V(\phi) + \frac{1}{4g_{SG}^2} F_{\mu\nu}^a F^{a\mu\nu}) \quad (2.4)$$

where $V(\phi)$ is a potential for the scalars, $F_{\mu\nu}^a = \nabla_{\mu} A_{\nu}^a - \nabla_{\nu} A_{\mu}^a + c_{bc}^a A_{\mu}^b A_{\nu}^c$, $\tilde{\nabla}_{\mu} \phi^I = \nabla_{\mu} \phi^I - i A_{\mu}^a t_a^{IJ} \phi^J$, and $[t_a, t_b] = c_{ab}^c t_c$. Including the gravitational part (see 2.29) we assume that the gravity-scalar subsystem possesses kink solutions of the form mentioned after (2.1).

¹We use Greek letters μ, ν, \dots to denote $(d+1)$ -dimensional coordinate indices and Latin letters i, j, \dots to denote d -dimensional coordinate indices.

²Some of the scalars ϕ^I to be introduced shortly will also be nonzero in the kink background and depend only on r , i.e., $\phi^I(x, r) = \phi^I(r)$.

To construct the Hamiltonian appropriate for radial evolution we decompose the metric as $g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu$ where n_μ is the unit normal to an $n-1$ dimensional surface of constant r and $n^\mu h_{\mu\nu} = 0$. Given a vector r^μ satisfying $r^\mu \nabla_\mu r = 1$ then n_μ may be decomposed as $n^\mu = \frac{1}{N}(r^\mu - N^\mu)$. Decomposing the vector potential as $A_\mu^a = n_\mu V^a/N + \tilde{A}_\mu^a$ where $n^\mu \tilde{A}_\mu^a = 0$ we find that

$$H = \int d^d x \sqrt{h} (N\mathcal{H} + N^\mu \mathcal{H}_\mu - V^a G_a) \quad (2.5)$$

where

$$\mathcal{H} = \mathcal{H}_A + \mathcal{H}_\phi \quad (2.6)$$

$$\mathcal{H}_A = \frac{g_{SG}^2}{2} \tilde{\pi}^{a\mu} \tilde{\pi}_\mu^a - \frac{1}{4g_{SG}^2} \tilde{F}_{\mu\nu}^a \tilde{F}^{a\mu\nu} \quad (2.7)$$

$$\mathcal{H}_\phi = \frac{1}{2} (\pi^I \pi_I - \tilde{D}_\mu \phi^I \tilde{D}^\mu \phi^I - V(\phi)) \quad (2.8)$$

$$\mathcal{H}_\nu = \tilde{\pi}^{a\mu} D_\nu \tilde{A}_\mu^a - D_\mu (\tilde{\pi}^{a\mu} \tilde{A}_\nu^a) + \pi^I D_\nu \phi^I \quad (2.9)$$

$$G_a = D_\mu \tilde{\pi}^{a\mu} - c_{ab} {}^c \tilde{A}_\mu^b \tilde{\pi}^{c\mu} + i t_a^{IJ} \phi^I \pi^J \quad (2.10)$$

$$\tilde{\pi}^{a\mu} = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \dot{\tilde{A}}_\mu^a}, \quad \pi^I = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \dot{\phi}^I} \quad (2.11)$$

and $\dot{\phi}^I := \mathcal{L}_r \phi^I$, $\dot{\tilde{A}}_\mu^a := \mathcal{L}_r \tilde{A}_\mu^a$, and D_μ is the $h_{\mu\nu}$ metric compatible connection. We will bring in the Einstein-Hilbert term for gravity momentarily but this will not affect the $V^a G_a$ part of the Hamiltonian, therefore we have the constraint $G_a(x) = 0$.

To arrive at the QFT Ward identities for the conserved currents $J^{ai}(x)$ and scalars $\mathcal{O}^I(x)$ dual to the supergravity vector $\tilde{A}_i^a(r, x)$ (we are now working in the radial gauge $N = 1$ and $N^i = 0$ so that $\tilde{A}_r^a(r, x) = 0$) and scalar $\phi^I(r, x)$ respectively now consists of two steps. The first is to recall from classical mechanics the Hamilton-Jacobi relations

$$\tilde{\pi}^{a\mu} = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \tilde{A}_\mu^a}, \quad \pi^I = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \phi^I} \quad (2.12)$$

where the action S has been evaluated on a solution to the equations of motion and is viewed as a functional of the boundary values of the fields $\tilde{A}_i^a(r_b, x)$ and $\phi^I(r_b, x)$ (the boundary being any surface of constant $r = r_b$). Substituting into the constraint arising from the non-Abelian gauge invariance (2.10) results in

$$\sqrt{h} D_i \left(\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \tilde{A}_i^a} \right) = c_{ab} {}^c \tilde{A}_i^b \frac{\delta S}{\delta \tilde{A}_i^c} - i t_a^{IJ} \phi^I \frac{\delta S}{\delta \phi^J}. \quad (2.13)$$

The second step to derive the QFT Ward identities is to use the correspondence (2.2). In particular, identifying³

$$\langle J^{ai}(x) \rangle_{\tilde{A}, \phi} = - \frac{\delta S}{\delta \tilde{A}_i^a(r_b, x)}, \quad \langle \mathcal{O}^I(x) \rangle_{\tilde{A}, \phi} = - \frac{\delta S}{\delta \phi^I(r_b, x)} \quad (2.14)$$

then we find the generating functional for Ward identities

$$\sqrt{h} D_i \left(\frac{1}{\sqrt{h}} \langle J^{ai}(x) \rangle_{\tilde{A}, \phi} \right) = c_{ab} {}^c \tilde{A}_i^b(r_b, x) \langle J^{ci}(x) \rangle_{\tilde{A}, \phi} - i t_a^{IJ} \phi^I(r_b, x) \langle \mathcal{O}^J(x) \rangle_{\tilde{A}, \phi} \quad (2.15)$$

where the subscript \tilde{A}, ϕ denotes that the expectation values are still being taken in the presence of the background sources. Evaluating on a kink solution⁴

$$h_{ij}(x, r) = \delta_{ij} e^{-2\lambda(r)}, \quad \phi^I(x, r) = \phi^I(r), \quad \tilde{A}_i^a = 0 \quad (2.16)$$

³The expectation values depend on the boundary radius r_b although we don't show this dependence explicitly.

⁴Note that this requires that $t_a^{IJ}(\phi^I \partial_r \phi^J - \phi^J \partial_r \phi^I) = 0$.

we find that in general not all currents will be conserved, i.e.,

$$\partial_i \langle J^{ai}(x) \rangle_\phi = -it_a^{IJ} \phi^I(r_b) \langle \mathcal{O}^J(x) \rangle_\phi, \quad (2.17)$$

where the subscript ϕ denotes that only some of the scalar sources are nonvanishing in the expectation values. From the supergravity perspective this nonconservation of the current arises from spontaneous symmetry breaking by the background solution. From the dual QFT perspective however it arises because the theory has been deformed away from its CFT critical point, explicitly breaking the global conservation law to a subgroup.

It is straightforward now to generate Ward identities for n -point correlators by functionally differentiating (2.15) with respect to the boundary values $\tilde{A}_i^a(r_b, x)$ and $\phi^I(r_b, x)$ (and of course $h_{ij}(r_b, x)$ as well). For example, differentiating (2.15) n times with respect to ϕ^I and evaluating on a kink solution (2.16) yields

$$\begin{aligned} \partial_i \langle J^{ai}(x) \mathcal{O}^{I_1}(x_1) \cdots \mathcal{O}^{I_n}(x_n) \rangle_\phi &= -i \sum_{i=1}^n t_a^{I_i J} \delta^{(d)}(x - x_i) \langle \mathcal{O}^J(x) \mathcal{O}^{I_1}(x_1) \cdots \widehat{\mathcal{O}^{I_i}(x_i)} \cdots \mathcal{O}^{I_n}(x_n) \rangle_\phi \\ &\quad - it_a^{IJ} \phi^I(r_b) \langle \mathcal{O}^J(x) \mathcal{O}^{I_1}(x_1) \cdots \mathcal{O}^{I_n}(x_n) \rangle_\phi \end{aligned} \quad (2.18)$$

where the notation $\widehat{\mathcal{O}^I}$ denotes that the operator is missing from the expectation value. For a pure AdS background, i.e., no scalars turned on, this reduces to the standard Ward identity which was checked in [22] to hold for $n = 2$ in the $\epsilon \ll 1$ limit where $r_b = -\ln \epsilon$. It was noted in [22] that the prescription originally given in [3] for computing 2-point functions did not yield results consistent with the Ward identities. [22] suggested another method for computing 2-point functions along the lines of [2] which involved solving the equations of motion subject to boundary conditions placed at a boundary $r = -\ln \epsilon$ for $\epsilon \ll 1$. In the Hamilton-Jacobi formalism used here this procedure comes out naturally.

In a similar way one may derive the Ward identity for the currents

$$\begin{aligned} \partial_i \langle J^{ai}(x) J^{a_1 i_1}(x_1) \cdots J^{a_n i_n}(x_n) \rangle_\phi &= \sum_{l=1}^n c^{aa_l c} \delta^{(d)}(x - x_l) \langle J^{ci}(x) J^{a_1 i_1}(x_1) \cdots \widehat{J^{a_l i_l}(x_l)} \cdots J^{a_n i_n}(x_n) \rangle_\phi \\ &\quad - it_a^{IJ} \phi^I(r_b) \langle \mathcal{O}^J(x) J^{a_1 i_1}(x_1) \cdots J^{a_n i_n}(x_n) \rangle_\phi \end{aligned} \quad (2.19)$$

where we have evaluated on the kink background (2.16). For the case of AdS background with no scalars turned on this Ward identity was checked in [22, 23] for $n = 2$ and $r_b = 1/\epsilon$.

Another case of interest is to add a Chern-Simons term for the vector field A_μ^a . For the particular case of type IIB supergravity on $AdS_5 \times S^5$ such an interaction term is produced and gives rise, as first pointed out by Witten, to the three point current correlator anomaly in the dual $\mathcal{N} = 4$ SYM theory. Witten further showed how to compute the anomaly in a very simple manner. Here we consider the Chern-Simons term from the Hamilton-Jacobi point of view and show that the anomaly again follows quite easily. Specifically consider the 5d Chern-Simons action

$$S_{CS} = \alpha \int d^5 x d_{abc} \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu^a \partial_\nu A_\lambda^b \partial_\rho A_\sigma^c \quad (2.20)$$

where $d_{abc} = Tr(\{t_a, t_b\}t_c)$. Repeating the above analysis with the action $S_\phi + S_A + S_{CS}$ gives rise to the Chern-Simons contribution to the gauge constraint

$$G_{a,CS} = -\alpha \frac{1}{\sqrt{h}} \epsilon^{ijkl} (d_{abc} \partial_i \tilde{A}_j^b \partial_k \tilde{A}_l^c + 2d_{bcd} c_{ea}^d \tilde{A}_i^b \partial_j \tilde{A}_k^c \tilde{A}_l^e) \quad (2.21)$$

where the 4-dimensional epsilon symbol satisfies $\epsilon^{\mu\nu\lambda\rho\sigma} = 5N\epsilon^{[\mu\nu\lambda\rho}n^{\sigma]}$. Making the substitution (2.12) for the conjugate momenta and functionally differentiating the above constraint two more times with respect to \tilde{A}_i^a gives rise to the Ward identity (2.19) for $n = 2$ plus the anomaly term

$$\frac{\partial}{\partial x^i} \langle J^{ai}(x) J^{bj}(y) J^{ck}(z) \rangle|_{anomaly} = -2\alpha d^{abc} \epsilon^{ijkl} \frac{\partial}{\partial x^i} \delta^{(d)}(x - y) \frac{\partial}{\partial x^l} \delta^{(d)}(x - z). \quad (2.22)$$

For a pure AdS background this agrees with the anomaly calculation in the dual $\mathcal{N} = 4$ SYM theory provided $\alpha = i(N^2 - 1)/(96\pi^2)$.

Including spinors and gravitinos is slightly more subtle than the cases discussed so far as the action principles are first order in derivatives. A consequence of this, see [26, 24], is that one cannot fix all components of the spinor at the boundary *and* demand regularity of the solution in the interior, but rather one must sacrifice half of the spinor

components at the boundary in order to keep regularity of the solution. To put it differently, half of the spinor components must be thought of as the fields, or “coordinates”, and the other half as the conjugate momenta. Which components to choose as the field variables is not arbitrary though. For example, the solution to the free massive spinor equations of motion on AdS gives rise to the relation [24]

$$\psi^+(\epsilon, k) = -i \frac{k_i \gamma_i}{k} \frac{K_{m-1/2}(k\epsilon)}{K_{m+1/2}(k\epsilon)} \psi^-(\epsilon, k) \quad (2.23)$$

where γ_i are the usual Dirac gamma matrices and the spinor components ψ^\pm are defined by $\psi^\pm = (1/2)(1 \pm \gamma_0)\psi$. For $m > 0$ we see that $\psi^+(\epsilon, k)$ goes to zero as $\epsilon \rightarrow 0$ for fixed $\psi^-(\epsilon, k)$. Fixing $\psi^+(\epsilon, k)$ instead would result in a divergent $\psi^-(\epsilon, k)$. Hence we conclude that ψ^- should be viewed as the field and $\bar{\psi}^+$ as the conjugate momentum, where we have defined $\bar{\psi}^\pm = (1/2)\bar{\psi}(1 \pm \gamma_0)$, and similarly $\bar{\psi}^-$ as the field and ψ^+ as the conjugate momentum. For $m < 0$ the opposite identifications are made. Another argument in favor of this identification was given in [27] where it was shown for $m > 0$ that $\psi^-(\epsilon, x)$ transforms under a local representation of the conformal group while $\psi^+(\epsilon, x)$ under a nonlocal representation. It follows that only $\psi^-(\epsilon, x)$ can act as a source for a quasi-primary operator in the dual CFT. Using this identification of fields and conjugate momenta it is straightforward to generalize the derivation of the Ward identities above to include spinor and gravitino fields.

B. Removing the cutoff

The derivation of the Ward identities presented so far has been in the presence of a finite cutoff. For massive scalars ($m_I^2 > 0$) this must be so as the space would not remain asymptotic to AdS_{d+1} in the $r \rightarrow \infty$ limit⁵. However, for tachyonic scalars, ie., those with masses satisfying $-d^2/4 < m_I^2 < 0$, we can remove the cutoff and rewrite the above Ward identities in terms of renormalized fields. One method that has been used [28,19] goes as follows. Evaluate the action at $r = -\ln \epsilon$ for $\epsilon \ll 1$. The scalars vanish as $\epsilon^{d-\Delta}$ (with Δ given in the previous footnote) in the $\epsilon \rightarrow 0$ limit, the vector goes to a constant in this limit, and the metric h_{ij} diverges as ϵ^{-2} , consequently the action diverges. Define the new fields

$$\tilde{\phi}^I(x_0, x) := (x_0)^{\Delta_I - d} \phi^I(x_0, x), \quad \tilde{h}_{ij}(x_0, x) := (x_0)^2 h_{ij}(x_0, x) \quad (2.24)$$

where $r = -\ln x_0$. More generally one can replace the multiplicative factor of x_0 in (2.24) by $f(x_0)$ so long as $f(x_0) \rightarrow x_0$ as $x_0 \rightarrow 0$. The point being that the new fields $\tilde{\phi}^I$ and \tilde{h}_{ij} approach constants as $x_0 \rightarrow 0$. Expressing the action in terms of the boundary values of the tilded fields allows one to isolate the divergences and remove them by adding counterterms. Furthermore the generating functional of Ward identities (2.15) when rewritten in terms of tilded fields takes the same form as in terms of the original fields, but now everything is finite and the $\epsilon \rightarrow 0$ limit can be taken.

Following [17] we may further rewrite the Ward identities as functions of the background kink solution evaluated at a finite radius (as opposed to leaving them as functions of the asymptotic values of the tilded fields defined above). This simply corresponds to choosing a different renormalization scale in the dual QFT. Define therefore the renormalized fields

$$\phi_R^I := \phi^I(x_0), \quad h_{ij,R} := h_{ij}(x_0) \quad (2.25)$$

for some x_0 . The fields ϕ_R^I and $h_{ij,R}$ can be expressed as functions of the asymptotic values of the fields $\tilde{\phi}^I(x_0)$ and $\tilde{h}_{ij}(x_0)$, and vice versa. In particular we have $\phi_0^I = \phi_0^I(\tilde{\phi}_R^J)$ where $\tilde{\phi}_0^J$ denotes the asymptotic value of $\tilde{\phi}^J(x_0)$. Under an infinitesimal global gauge transformation this relation yields

$$\epsilon^a t_a^{IJ} \phi_0^J = \phi_0^I(\epsilon^a t^{JK} \tilde{\phi}_R^K) \approx \epsilon^a \frac{\partial \phi_0^I}{\partial \tilde{\phi}_R^J} t_a^{JK} \phi_R^K. \quad (2.26)$$

Defining the renormalized operators \mathcal{O}_R^I by

⁵Recall that a scalar field with mass m_I in an AdS_{d+1} background behaves near the AdS_{d+1} boundary as $x_0^{d-\Delta}$ where $\Delta = (1/2)(d + \sqrt{d^2 + 2m_I^2})$ and $r = -\ln x_0$.

$$\mathcal{O}_R^I := \frac{\partial \tilde{\phi}_0^J}{\partial \phi_R^I} \mathcal{O}^J \quad (2.27)$$

and using the relation (2.26) it follows that the Ward identity (2.18) (by which we really mean (2.18) expressed in terms of the tilded fields), and more generally any Ward identity involving the scalars \mathcal{O}^I and currents J^{ai} , takes the same form, i.e.,

$$\begin{aligned} \partial_i \langle J^{ai}(x) \mathcal{O}_R^{I_1}(x_1) \cdots \mathcal{O}_R^{I_n}(x_n) \rangle_\phi &= -i \sum_{i=1}^n t_a^{I_i J} \delta^{(d)}(x - x_i) \langle \mathcal{O}_R^J(x) \mathcal{O}_R^{I_1}(x_1) \cdots \hat{\mathcal{O}}_R^{I_i}(x_i) \cdots \mathcal{O}_R^{I_n}(x_n) \rangle_\phi \\ &\quad - i t_a^{IJ} \phi_R^I \langle \mathcal{O}_R^J(x) \mathcal{O}_R^{I_1}(x_1) \cdots \mathcal{O}_R^{I_n}(x_n) \rangle_\phi \end{aligned} \quad (2.28)$$

where the expectation values are now expressed as functions of ϕ_R^I .

C. Inclusion of the metric:

We can further derive QFT Ward identities associated with translation invariance by looking at the constraint arising from varying the shift vector N^μ . We must first however include the gravitational part of the action given by the Einstein-Hilbert Lagrangian

$$S_{EH} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{g} R. \quad (2.29)$$

The Hamiltonian for this system again takes the same form as in (2.5) with vanishing contribution to the constraint G_a while

$$\mathcal{H}_g = -\frac{1}{2\kappa^2} (\pi^{\mu\nu} \pi_{\mu\nu} - \frac{1}{n-2} \pi^2 + \tilde{R}) \quad (2.30)$$

$$(\mathcal{H}_g)_\nu = -\frac{1}{\kappa^2} D_\mu \pi^\mu{}_\nu \quad (2.31)$$

$$\frac{1}{2\kappa^2} \pi^{\mu\nu} = \frac{1}{\sqrt{h}} \frac{\delta S_{EH}}{\delta \dot{h}_{\mu\nu}} \quad (2.32)$$

where \tilde{R} is the Ricci scalar associated with $h_{\mu\nu}$. Combining the vector constraint for the Einstein-Hilbert action (2.31) with that for the scalar-vector system (2.9) and replacing momenta as in (2.12) for the scalars and vectors and

$$\frac{1}{2\kappa^2} \pi^{ij} = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h_{ij}} \quad (2.33)$$

for the metric (evaluating in the radial gauge) results in

$$-2\sqrt{h} D_j \left(\frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^i{}_j} \right) + \partial_i \phi^I \frac{\delta S}{\delta \phi^I} + D_i \tilde{A}_j^a \frac{\delta S}{\delta \tilde{A}^{aj}} - \sqrt{h} D_j \left(\frac{1}{\sqrt{h}} \frac{\delta S}{\delta \tilde{A}_j^a} \tilde{A}_i^a \right) = 0. \quad (2.34)$$

Assuming that the dual QFT stress tensor couples to the boundary value of the metric as $\frac{1}{2} \int d^d x T_{ij} h^{ij}$, then evaluating (2.34) on a kink background implies that the stress tensor is conserved, i.e., $\partial_i \langle T^{ij}(x) \rangle_\phi = 0$. To derive the Ward identities we need only functionally differentiate as before, eg., differentiating (2.34) n times with respect to $\phi^{I_i}(x_i)$ and evaluating on the kink background results in

$$\frac{\partial}{\partial x^j} \langle T^j{}_i(x) \mathcal{O}^{I_1}(x_1) \cdots \mathcal{O}^{I_n}(x_n) \rangle = \sum_{k=1}^n \frac{\partial}{\partial x^i} \delta^{(d)}(x - x_k) \langle \mathcal{O}^{I_1}(x_1) \cdots \mathcal{O}^{I_k}(x_k) \cdots \mathcal{O}^{I_n}(x_n) \rangle. \quad (2.35)$$

Differentiating instead with respect to the vector one finds instead

$$\begin{aligned} \frac{\partial}{\partial x^j} \langle T^i{}_j(x) J^{a_1 i_1}(x_1) \cdots J^{a_n i_n}(x_n) \rangle &= \sum_{k=1}^n \left(\frac{\partial}{\partial x^i} \delta^{(d)}(x - x_k) \langle J^{a_1 i_1}(x_1) \cdots J^{a_k i_k}(x_k) \cdots J^{a_n i_n}(x_n) \rangle \right. \\ &\quad \left. - \frac{\partial}{\partial x^j} (\delta^{(d)}(x - x_k) \delta_i^{j k} \langle J^{a_1 i_1}(x_1) \cdots J^{a_k j}(x_k) \cdots J^{a_n i_n}(x_n) \rangle) \right). \end{aligned} \quad (2.36)$$

To derive the Ward identities for the renormalized fields one can follow the procedure outlined previously.

We have shown that Ward identities in the quantum field theory dual of a string theory on an asymptotically AdS space follow quite naturally from the supergravity constraint equations. Specifically associated to any gauge symmetry of the bulk supergravity theory there is a corresponding constraint equation. The generating functional of QFT Ward identities follows after rewriting this constraint equation in Hamilton-Jacobi form. We have considered in particular the examples of a non-Abelian gauge invariance and diffeomorphism invariance in the bulk and have computed the corresponding Ward identity generating functionals.

It is not surprising that there is a corresponding Lagrangian approach to deriving these Ward identities as well. Namely varying the AdS/CFT relation (2.2) under a supergravity gauge transformation leaves the supergravity side of the relation invariant (up to possible anomalies, one case of which was discussed in section II A). The dual QFT expectation value however is not invariant and gives rise to a generating functional for the corresponding Ward identities. This argument was already implicit in Witten's computation of the R -current anomaly for 4d, $\mathcal{N} = 4$ super Yang-Mills theory and more recently was used explicitly in [19] to derive the generating functional (2.34) (without the vector field). One advantage in the approach used here to derive QFT Ward identities is that it sheds some light on the inverse problem of reconstructing the bulk spacetime from the dual QFT. Specifically in the approach of this paper one can see more explicitly how the QFT Ward identities correspond to the bulk supergravity constraint equations. One final comment is that it would be interesting to find a Lagrangian approach to deriving the QFT renormalization group equations dual to the Hamilton-Jacobi approach used in [17], which one would naturally expect must exist especially in light of the dual approaches to deriving Ward identities.

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- [1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.
 - [2] S. Gubser, I. Klebanov and A. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, Phys.Lett.B428 (1998) 105, hep-th/9802109.
 - [3] E. Witten, *Anti-de Sitter Space and Holography*, Adv.Theor.Math.Phys.2 (1998) 253, hep-th/9802150.
 - [4] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, *Large N Field Theories, String Theory and Gravity*, hep-th/9905111.
 - [5] L. Susskind and E. Witten, *The Holographic Bound in Anti-de Sitter Space*, hep-th/9805114.
 - [6] A. W. Peet and J. Polchinski, *UV/IR Relations in AdS Dynamics*, Phys. Rev. **D59** (1999) 065011.
 - [7] E.T. Akhmedov, *A Remark on the AdS/CFT Correspondence and the Renormalization Group Flow*, Phys. Lett. **B442** (1998) 152, hep-th/9806217.
 - [8] E. Alvarez and C. Gómez, *Geometric Holography, the Renormalization Group and the c -Theorem*, Nucl. Phys. **B541** (1999) 441, hep-th/9807226.
 - [9] D. Z. Freedman, S. S. Gubser, K. Pilch, N. P. Warner, *Renormalization Group Flows from Holography-Supersymmetry and a c -Theorem*, hep-th/9906194.
 - [10] L. Girardello, M. Petrini, M. Porrati, A. Zaffaroni, *The Supergravity Dual of $N = 1$ Super Yang-Mills Theory*, Nucl. Phys. **B569** (2000) 451, hep-th/9909047.
 - [11] L. Girardello, M. Petrini, M. Porrati, A. Zaffaroni, *Novel Local CFT and Exact Results on Perturbations of $N=4$ Super Yang Mills from AdS Dynamics*, JHEP 9812 (1998) 022, hep-th/9810126.
 - [12] M. Porrati, A. Starinets, *RG Fixed Points in Supergravity Duals of 4-d Field Theory and Asymptotically AdS Spaces*, Phys.Lett. **B454** (1999) 77, hep-th/9903241.
 - [13] K. Skenderis, P.K. Townsend, *Gravitational Stability and Renormalization-Group Flow*, Phys. Lett. **B468** (1999) 46, hep-th/9909070.
 - [14] O. DeWolfe, D.Z. Freedman, S.S. Gubser, A. Karch, *Modelling the fifth dimension with scalars and gravity*, hep-th/9909134.
 - [15] O. DeWolfe and D. Z. Freedman, *Notes on Fluctuations and Correlation Functions in Holographic Renormalization Group Flows*, hep-th/0002226.

- [16] V. Balasubramanian, P. Kraus, *Space-time and the Holographic Renormalization Group*, Phys.Rev.Lett 83 (1999) 3605, hep-th/9903190.
- [17] J. de Boer, E. Verlinde, and H. Verlinde, *On the Holographic Renormalization Group*, hep-th/9912012.
- [18] M. Li, *A note on relation between holographic RG equation and Polchinski's RG equation*, hep-th/0001193.
- [19] S. de Haro, K. Skenderis, and S.N. Solodukhin, *Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence*, hep-th/0002230.
- [20] V. Balasubramanian, E. Gimon, and D. Minic, *Consistency Conditions for Holographic Duality*, hep-th/0003147.
- [21] G. Lifschytz and V. Periwal, *Schwinger-Dyson= Wheeler-DeWitt: gauge theory observables as bulk operators*, hep-th/0003179.
- [22] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation Functions in the CFT(D)/ADS(D+1) Correspondence*, Nucl. Phys. **B546** (1999) 96, hep-th/9804058.
- [23] G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, *R-current correlators in N=4 Super Yang-Mills theory from anti-de Sitter supergravity*, Nucl. Phys. **B540** (1999) 247, hep-th/9805105.
- [24] W. Mück and K.S. Viswanathan, *Conformal Field Theory Correlators from Classical Field Theory on Anti-de Sitter Space II. Vector and Spinor Fields*, Phys. Rev. **D58** (1998) 106006 [hep-th/9805145].
- [25] H. Liu and A.A. Tseytlin, *D=4 Super Yang Mills, D=5 gauged supergravity and D=4 conformal supergravity*, Nucl. Phys. **B533** (1999) 88, hep-th/9804083.
- [26] M. Henningson and K. Sfetsos, *Spinors and the ADS/CFT Correspondence*, Phys. Lett. **B431** (1998) 63, hep-th/9803251.
- [27] G.E. Arutyunov and S.A. Frolov, *On the origin of supergravity boundary terms in the AdS/CFT correspondence*, hep-th/9806216.
- [28] I.R. Klebanov and E. Witten, *AdS/CFT correspondence and symmetry breaking*, Nucl. Phys. **B556** (1999) 89, hep-th/9905104.